Consequences of the Riemann Hypothesis Paper

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1 Overview

In this paper, I will give an introduction to the Riemann function, the Riemann hypothesis, prime counting, the Chebychev functions, and related concepts. We will explore how they relate to prime numbers. We will mostly be looking to associated functions which relate to prime counting and the zeta function such as the Chebyshev functions.

2 Introduction

The Riemann Hypothesis is not only interesting because of the fascinating math, but also the history behind it. Bernard Riemann conjectured the Riemann Hypothesis. In his original paper, Riemann wrote, "Of course one would wish for a rigorous proof here." This statement would go on to be ironic as over 160 years later, the Riemann Hypothesis still has not been proven. It says something fundamental about the distribution of primes, so many proofs assume it as a given. In 1900, the great mathematician Dave Hilbert gave what he deemed the most important problems in math, that he thought would define the next century of math. His 8th problem was the Riemann Hypothesis, showing how important he considered it. In 2000, the Clay Institute offered a $1 million prize for a proof of the Riemann Hypothesis

3 Prime Counting

Definition 1 *$\pi(n)$ is the number of primes less than n*

The distribution of the primes is very important to mathematics, so over the course of this paper we will see several functions which relate to π and help us approximate it.

3.1 The Prime Number Theorem

Primes are a very enigmatic and important concept in mathematics, and so the prime number theorem says something about the distribution of primes.

Definition 2 *prime number theorem $\pi(n) \approx \frac{x}{\log(x)}$*

The most interesting fact about the Prime Number Theorem is how other theorems relate to it, which we shall see.

4 The Zeta Function

The statement of the Riemann Hypothesis centers around the zeta function.

Definition 3

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

An alternate expression is to factorize each number in the sum as its prime factorization. Doing this gives

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} \cdots \implies (1 - \frac{1}{2})\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} \cdots \implies (1 - \frac{1}{3})(1 - \frac{1}{2})\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} \cdots$$

. If we continue this we get

Definition 4 *product representation of the Zeta function*

$$\zeta(s) = \Pi_{p|prime} \frac{1}{1 - p^{-1}}$$

As an aside, this gives an alternate proof that there are infinite primes. Since $\zeta(1)$ diverges, there must be infinitely many primes.

Now that we have a definition of the Riemann Zeta function, we should prove some properties that are true about it.

Theorem 1 *if $Re(s) > 1$ then $\zeta(s) \neq 0$*

If $\sum_n |a_n|$ converges then the product

$$\prod_{n=1}^{\infty}(1+a_n)$$

converges and in this case the product converges to 0 if and only if one of the factors is 0. In this case we have

$$\begin{aligned}(1-p^{-s})^{-1} &= \left(\frac{p^s-1}{p^s}\right)^{-1} \\ &= \frac{p^s}{p^s-1} \\ &= 1+\frac{1}{p^s-1}\end{aligned}$$

so apply the proposition for $a_n = (p^s-1)^{-1}$ to see that the product converges for $Re(s) > 1$. Then we know $(1-p^{-s})^{-1} \neq 0$ for all primes p and so using the identity and the second statement in the proposition $\zeta(s) \neq 0$ for all $s \in C$ with $Re(s) > 1$.

4.1 Continuation

Note that with this sum representation, $\zeta(s)$ is only defined for $Re(s) > 1$, so how can we continue it to be defined for all $C\backslash 0$. To do this we will define another function.

Definition 5 *The Direchlet eta function is*

$$\eta(s) = \sum_{n=1}^{\infty}\frac{-1^{n-1}}{n^s} = (1-2^{1-s})\zeta(s)$$

If you want the analytic continuation of the zeta function to the zone where all the non-trivial zeros have been found so far, you can do as follows:

$$(1)\ \zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$$

$$(2)\ \sum_{n=1}^{\infty}\frac{2}{(2n)^s} = \frac{1}{2^{s-1}}\zeta(s)$$

Now, subtract (2) from (1):

$$\left(1-\frac{1}{2^{s-1}}\right)\zeta(s) = \frac{1}{1^s}-\frac{1}{2^s}+\frac{1}{3^s}-\ldots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^s} =: \eta(s) \implies \implies \zeta(s) = \left(1-2^{1-s}\right)^{-1}\eta(s)$$

As an exercise to the reader, prove that this expression is analytic on.

$$1 \neq Re(s) > 0\,.$$

4.2 the Riemann Hypothesis

Theorem 2 *The Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part* $\frac{1}{2}$. $\zeta(s) = 0 \implies Re(s) = \frac{1}{2}$

The Riemann Hypothesis implies the best possible bound on the prime number theorem, which is why it is so important, but we will not prove that in this paper.

5 Chebyshev functions

In order to define the Chebychev functions, we must first define the Von Mangoldt function

Definition 6 *the Von Mangoldt function is*

$$\Lambda(n) = \begin{cases}\log p & \textit{if } n = p^k \textit{ for some prime } p \textit{ and integer } k \geq 1, \\ 0 & \textit{otherwise.}\end{cases}$$

Then, using this function we can define the Chebyshev function as

Definition 7 *The Chebyshev* ψ *function is the partial sum of* $\Lambda(n)$*, i.e.*

$$\psi(x) = \sum_{n\leq x}\Lambda(n)$$

The final function which relates to the Chebchev ψ function is the closest related ψ_1 function which is defined as

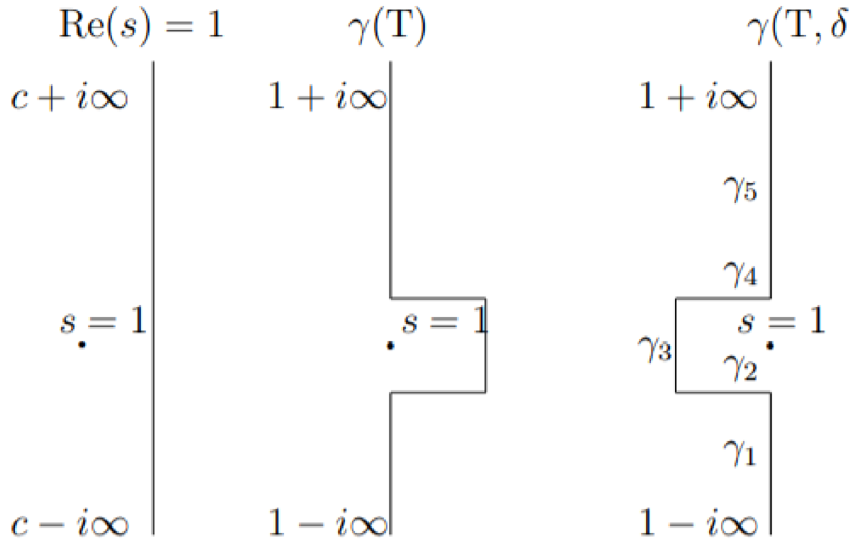

Figure 1: the paths we will use for this proof

Definition 8 *The Chebyshev ψ_1 function is $\psi_1(x) = \int_1^x \psi(u)du$.*

The most interesting property about the Chebyshev function is how it relates to the prime number theorem.
Now that we have defined the Chebyshev Function, let us prove a property about the Chebyshev function.

Theorem 3 *An approximation of the asymptotic behavior of the Chebyshev function is*

$$\psi(x) \approx x$$

I will not prove this theorem, but it is fairly easy to do, so I will leave it as an exercise to the reader.

6 Proof

I would like to end the paper of a proof as some food for thought. I am going to prove that

Theorem 4 $\psi_1 \to \frac{x^2}{2}$ *as* $x \to \infty$

Before we can, we need a preliminary lemma,

Theorem 5 *An alternate expression of ψ_1 is*

$$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^s+1}{s(s+1)}(\frac{-\zeta'(s)}{\zeta(s)})ds$$

We will not prove this currently, but it can be left as an exercise.

We know

$$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{xs+1}{s(s+1)}(-\frac{\zeta'(s)}{\zeta(s)})ds.$$

In this integral, we integrate on the line $Re(s) = c$. We will now transform this to integrate over $Re(s) = 1$. When $c > 1$. For now we assume that $x \geq 2$. Let

$$F(s) = \frac{x^s+1}{s(s+1)}(-\frac{\zeta'(s)}{\zeta(s)})$$

. We define the paths below using T to determine the size of the "box" containing the point s. We will use this image to define the paths for the integration.

An application of Cauchy's theorem shows that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)ds = \frac{1}{2\pi i}\int_{\gamma(T)} F(s)ds$$

Now we need to show the equivalence of the integral over $\gamma(T,\delta)$. For any T, we can pick a $\delta > 0$ such that the zeta function has no zeros when $|Im(s)| \leq T$ and $1-\delta \leq Re(s) \leq 1$ as the zeta function has no zeros on $Re(s) = 1$. F(s) has a simple pole at s = 1. The residue of F(s) at $s = 1$ is $\frac{x^2}{2}$. Using this fact, we can find that

$$\frac{1}{2\pi i}\int_{\gamma}(T)F(s)ds = \frac{x^2}{2} + \frac{1}{2\pi i}\int_{\gamma}(T,\delta)\frac{x^{s+1}}{s(s+1)}F(s)ds$$

If we let $s \in \gamma_1$, we see that

$$|x^{1+s}|\int_{\gamma_1} F(s)ds| \leq \frac{\epsilon}{2}x^2| = x^2$$

. Although we will not prove that it exists, we will assume that there is a C such that,

$$|\int_{\gamma_1} F(s)ds| \le Cx^2 \int_T^\infty \frac{|t|^{1/2}}{t^2}dt$$

. Because this converges, we can find T that is big enough so that

$$|\int_{\gamma_1} F(s)ds| \le \frac{\epsilon}{2}x^2$$

and

$$|\int_{\gamma_5} F(s)ds| \le \frac{\epsilon}{2}x^2.$$

We fix a T to satisfy this and pick a δ small enough as well. Since, on γ_3, we have $|x^{1+s}| = x^{1+1-\delta} = x^{2-\delta}$, we can find a C' such that $|\int_{\gamma_3} F(s)ds| \le C'x^{2-\delta}$. We can also approximate γ_2 and γ_4 by

$$|\int_{\gamma_2} F(s)ds| \le C'' \int_{1-\delta}^1 x^{1+\sigma}d\sigma \le C''\frac{x^2}{\log x}$$

. Thus, we now have

$$|\psi_1(x) - \frac{x^2}{2}| \le \epsilon x^2 + C'x^{2-\delta} + C''\frac{x^2}{\log x}$$

. Thus, we have $|\frac{2\psi_1(x)}{x^2} - 1| \le 4\epsilon$ .Because we have bounded this expression, we can say that the theorem has been proved.

What does this proof show? It connects the Chebychev functions to the Riemann zeta function.

7 Conclusion

Despite the long time since Riemann published his Hypothesis, mathematicians still believe we are not close to solving it. Some partial progress has been made on it, which takes the form of equivalent statements. The Riemann Hypothesis has equivalent statements in harmonic numbers, the Mobius function, the von Mangoldt function, and many others. Equivalent expressions open up different methods of examining the Riemann Hypothesis. For example, the Riemann Hypothesis is true if $\sigma(n) \le H_n + e^{H_n}\log(H_n)$ for every $n \ge 1$, where $\sigma(n)$ is the sum of divisors function and H_n is the nth harmonic number. Such alternate approaches allow mathematicians to employ the tools of other fields, such as number theory, and not just complex analysis.

References

[1] Samuel Broadbent, Habiba Kadiri, Allysa Lumley, Nathan Ng, and Kirsten Wilk. Sharper bounds for the chebyshev function $\theta(x)$. *arXiv preprint arXiv:2002.11068*, 2020.

[2] Anatoly A Karatsuba and Sergei Mikhailovich Voronin. *The Riemann zeta-function.* de Gruyter, 1992.

[3] N Costa Pereira. Estimates for the chebyshev function ψ(x)-θ(x). *Mathematics of computation*, pages 211–221, 1985.

[4] Edward Charles Titchmarsh, David Rodney Heath-Brown, Edward Charles Titchmarsh Titchmarsh, et al. *The theory of the Riemann zeta-function.* Oxford University Press, 1986.

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