PERRON FROBENIUS THEOREM EULER CIRCLE PAPER

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CONTENTS

1. INTRODUCTION

When looking at matrices, forcing ”nice” properties such as being square can lead to other interesting results. In 1907, Perron and Frobenius realised several properties of real square matrices that were unexpected and useful. This theorem is used in several areas, such as ergodic Markov chains. In this paper we will look at the Perron-Frobenius theorem, prove some of its statements, and explore into some applications.

2. BACKGROUND

Before we can state the Perron-Frobenius theorem, we need a few definitions. These definitions mostly come from linear algebra, because the Perron-Frobenius theorem deals with that subject.

Definition 1. *If A is an nx n matrix and λ is a scalar, the λ-eigenspace is the set of all vectors $\overrightarrow{v}$ $\in \mathbf{R}^n$ such that $A\overrightarrow{v} = \lambda\overrightarrow{v}$. So, the nonzero vectors in Eλ are exactly the eigenvectors of A with eigenvalue λ.*

We can think of the eigenspace as all the vectors that are the same when multiplying by the matrix and the scalar.

Definition 2. *The dimension of the eigenspace is called the geometric multiplicity of λ. The algebraic multiplicity of an eigenvalue is the multiplicity of the root.*

Definition 3. *The spectral radius ρ(A) of a matrix is the max of the absolute values of the eigenvalues of A.*

2.1. The Perron-Frobenius theorem.

Now that we have defined these terms, we are ready to state the Perron-Frobenius theorem. I will state the version for positive matrices, although there does exist a version for non-negative matrices.

theorem 4 (Perron-Frobenius)**.** *Let $A = (a_{ij})_{1\leq i,j\leq n}$ be an $n \times n$ square matrix such that $a_{ij} > 0$ for all i, j.*

- *There is a positive number r, called the* Perron-Frobenius eigenvalue *of A, such that r is an eigenvalue of A, and for every other eigenvalue λ of A, $|\lambda| < r$.*

- *The eigenspace corresponding to the Perron-Frobenius eigenvalue is 1-dimensional.*
- *There exists an eigenvector v, called the* Perron-Frobenius eigenvector*, for the Perron-Frobenius eigenvalue such that all the entries of v are positive.*
- *The only nonnegative eigenvectors of A are multiples of the Perron-Frobenius eigenvector.*

3. Proof of the Perron-Frobenius theorem

3.1. **Introduction.** The proof of the Perron-Frobenius theorem can be somewhat long, but it is manageable, and I would say it can lead to a deeper understanding the Perron-Frobenius theorem.

I will split up each statement of the Perron-Frobenius theorem, and prove each of them separately.

3.2. **Perron-Frobenius Eigenvalue.** First I will prove there is a number r which is the Perron-Frobenius Eigenvalue.

First we will show there exists a postive eigenvalue. The trace of the matrix is positive, and since the trace can be expressed as the sum of the eigenvalues then there is a postitive eigenvalue.

Since the matrix is positive, there must exist an eigenvalue which is larger than all other eigenvalues.

3.3. **Eigenspace is 1 Dimensional.** I will prove that the Eigenspace is 1 dimensional. The dimension is also sometimes called the geometric multiplicity. I will accomplish this by a proof by contradiction.

Assume that v is an eigenvector and v' is a linearly independent eigenvector of r. v' is a linearly independent eigenvector of the eigenvalue ρ(A). We can assume that v' is real; otherwise we take real and imaginary parts, and the parts are still eigenvectors, because A and ρ(A) are real. One of them must be linearly independent of v. Let c $>$ 0 be chosen so that v - cv' is non-negative and at least one entry is zero. It is not the zero vector, because v, v' are linearly independent. However $v - cv' = \frac{A[v-cv']}{\rho(A)} > 0$ We chose c so that at least one entry was zero, meaning a contradiction. Thus, there cannot be two linearly independent eigenvectors, so ρ(A) has geometric multiplicity 1.

3.4. **Multiples of the Perron-Frobenius Eigenvector.** I will prove that the only eignenvectors are multiples of the Perron-Frobenius eigenvector. This step mostly follows from the fact that the Perron-Frobenius eigenspace is 1 dimensional Let λ be an eigenvalue and y be an positive eigenvector. Let x be the Perron-Frobenius eigenvector, then $rxTy = (x^T A)y = x^T(Ay) = \lambda x^T y$, and $x^T y > 0$, which means that that we has r = λ. Because the eigenspace is one-dimensional, the eigenvector y is a multiple of the Perron–Frobenius eigenvector.

4. Applications

One application of the Perron-Frobenius Theorem is applying it to Markov chains.

4.1. **Stationary Distribution.** The classic example is proving that every Markov chain has a unique stationary distribution.

Let π denote a Perron-Frobenius eigenvector of P, with $\pi \geq 0$ and $\mathbf{1}^T\pi$ = 1 because P π = π, π corresponds to a stationary distribution. of the Markov chain. Now, let P be regular, which means for some k,$P^k > 0$. Since $(P^k)_{ij}$ is $\mathbf{P}(X_{t+k} = i|X_t = j)$, this means there is a positive probability of going from any state to any other in k steps since P is regular, there is a unique invariant distribution π, which satisfies $\pi > 0$ the eigenvalue 1 is simple and dominant, so we have $p_t \to \pi$, no matter what the initial distribution p_0 in other words: the distribution of a regular Markov chain always converges to the unique invariant distribution.

4.2. **Pagerank.** A perhaps more real world example of the Perron-Frobenius theorem might be in pagerank, the algorithm used to rank pages in search engines. It can be thought of as a random walk on websites, where at each step the walker clicks on a random link on a site.

If there are n sites, then the adjacency matrix is a n x n matrix with entries $A_{ij} = 1$ if there exists a link from $a_j$to a_i . If we divide each column by the number of 1 in that column, we obtain a Markov matrix A which is called the normalized web matrix. Define the matrix E which satisfies $E_{ij} = \frac{1}{n}$ for all i, j.

The Google matrix is then $dA + (1d)E$, where $0 < d < 1$ is a parameter called damping factor.

The reason this example relates to the Perron-Frobenius theorem is that the Perron-Frobenius eigenvector of A scaled so that the largest value is 10 , and it is called page rank of the damping factor d. Although this example is somewhat more tangentially related, it is nice to know this theorem is useful in some unexpected places.

5. Conclusion

The Perron-Frobenius theorem is a classic example of how just from restricting a few properties, many other consequences follow.

If you are interested in learning more about the Perron-Frobenius theorem, here are some resources which go into further depth on the subject: " Perron-Frobenius Theorem for Nonnegative Tensors" by Chang, Kung-Ching, Kelly Pearson, and Tan Zhang and "The Perron–Frobenius Theorem and the Ranking of Football Teams" by Keener, James

References

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