SETS, FUNCTIONS, AND THE CONTINUUM HYPOTHESIS (CHAPTER 19 IN PFTB)

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1. INTRODUCTION

In 1910, mathematicians Bertrand Russel and Alfred Whitehead wrote *Principia Mathematica* in order to formalize all of mathematics using rigorous definitions and proofs. The tool that they used to accomplish this monumental feat was set theory. In this paper, we will look at some of the most interesting results of set theory. It will be based on chapter 19 of *Proofs from the Book* "Sets, functions, and the continuum hypothesis" by Aigner and Ziegler, although we will present different proofs.

2. BASIC SET THEORY, CARDINALITY, AND BIJECTIONS

One of the most basic concepts of a set is its cardinality.

Definition 1. *Given a set, the* ***cardinality*** *of that set is loosely defined as the size of that set.*

For a finite set, the cardinality is the number of elements in the set. For example, the cardinality of $\{4, 5, 6\}$ is 3. We write the cardinality of X with $|X|$, so for example $|\{4, 5, 6\}| = 3$. The traditional way of proving the cardinality of a set is with a bijection

Definition 2. *A* ***bijection*** $\varphi : A \to B$ *is a mapping between 2 sets such that it is*

(1) *Surjective: for all $a \in A$, there exists $b \in B$ such that $\varphi(a) = b$,*
(2) *Injective: for all $b \in B$, there exists an $a \in A$ such that $\varphi^{-1}(b) = a$*

In short, a bijection is just a correspondence between 2 sets. Injective and surjective each give an inequality. If $\varphi : A \to B$ is injective, then $|A| \leq |B|$, and if it is surjective, then $|A| \geq |B|$.

Using bijections, we can prove properties about the cardinality of sets.

Theorem 3. *If there is a bijection between 2 sets they have the same cardinality.*

For example, consider the sets $\{1, 2, 3\}$ and $\{4, 5, 6\}$. They have the same cardinality because there exists the bijection $\varphi : 1 \to 4, 2 \to 5, 3 \to 6$. This theorem is important because it works even for infinite sets.

3. COUNTABILITY AND BIJECTIONS

Once we have the power of bijections, we can make statements about the size of infinite sets. First, the concept of Countability.

Definition 4. *A set is* ***countable*** *if it has the same cardinality as the set of natural numbers $\mathbb{N}$.*

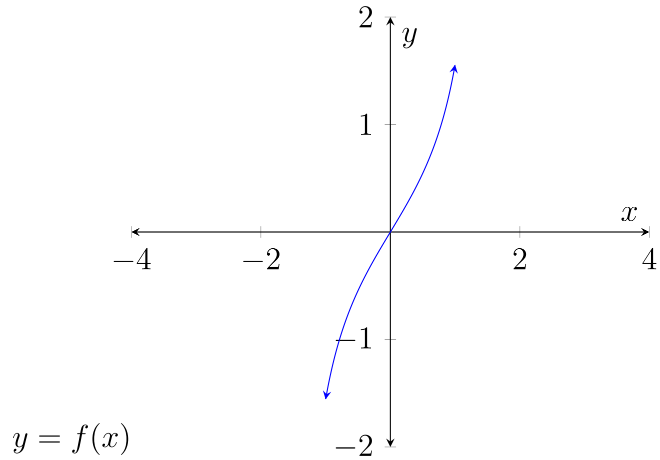
Using Theorem 3, we can also say a countable set has a bijection to $\mathbb{N}$ For example, the set of even numbers $\{2, 4, 8, \cdots\}$ because we can construct the bijection $\varphi : x \to 2x$. Another example is the set of all integers $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$ is countable due to the bijection

$$\varphi(x) = \begin{cases} 2x & x \geq 0 \\ -2x - 1 & x < 0 \end{cases}$$

Example 5. $(0, 1) \to (-\infty, \infty)$ *is a bijection*

Proof. Consider the function $f(x) = \tan(\pi(x - \frac{1}{2}))$. This function is a bijection between $(0, 1)$ and $(-\infty, \infty)$. To see why, look at 1. $\square$

FIGURE 1. An example of a bijection between $(0, 1)$ and $(-\infty, \infty)$



Example 6. $\mathcal{P}(\mathbb{N}) \to 3^{\mathbb{N}}$ *is a bijection*

For notation, $\mathcal{P}(\mathbb{N})$ is the power set of natural numbers, and $3^{\mathbb{N}}$ is the set of ternary strings.

Proof. The first step is to have a bijection from $f : \mathcal{P}(\mathbb{N}) \to 2^{\mathbb{N}}$. For each element $a \in \mathbb{N}$ and every subset $S \subset N$, we can define a function f:

$f(a) = 0$ if $a \notin S$

$f(a) = 1$ if $a \in S$

This proves a bijection between $2^{\mathbb{N}}$ and $\mathcal{P}(\mathbb{N})$ exists and therefore, $|2^{\mathbb{N}}| = \mathcal{P}(\mathbb{N})$. Next, define a bijection between $2^{\mathbb{N}}$ and $3^{\mathbb{N}}$;for this map $g : 0 \to 1, 1 \to 01, 2 \to 00$ so for instance $0121 \cdots$ gets mapped to $1010001 \cdots$ its easy to see this is a bijection. $\square$

Theorem 7. $\mathbb{R}$ *is uncountable*

The most common proof that $\mathbb{R}$ is uncountable is Cantor's diagonalization argument (presented in PFTB). I will give a different proof.

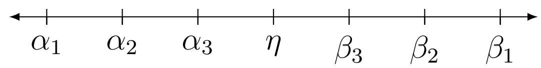
Proof. Imagine we had a list of all real numbers

$$l = \{r_1, r_2, r_3, r_4 \cdots\}$$

Choose 2 arbitrary real numbers, call the smaller α_1 and the larger β_1. Scan the list of reals from left to right until the first pair of reals between α_1 and β_1, call these numbers α_2 and β_2. We use this process to create 2 sequences as in figure 2. Take the limit of these sequences, call them α_∞ and β_∞. These limits exist because the sequences of α and β are monotonic and bounded. If $\alpha_\infty \neq \beta_\infty$, then there would be some number between them, for example $\frac{\alpha_\infty + \beta_\infty}{2}$, which is not on the list of reals, a contradiction. If $\alpha_\infty = \beta_\infty$, then call $\eta = \alpha_\infty = \beta_\infty$. Since η is a real, it must have an index on the list of reals, but it cannot have a finite index because an infinite number of α and β terms preceded in on the list, which means η cannot have a finite index on the list of reals, a contradiction. In either case, we cannot have a list of all reals, so $\mathbb{R}$ must be uncountable. $\square$

In other words, we have shown that $|\mathbb{N}| < |\mathbb{R}|$ because there is an injective function (e.g $\varphi : x \to x$) but we have shown there does not exist any bijective function.

FIGURE 2. The series of α and β



4. Cardinal and Ordinal Numbers

Cardinal and ordinal numbers test our intuition about what a number is, and they both tackle infinity in different ways. We will look at both.

Definition 8. *Cardinals*

The cardinals are a generalization of the natural numbers. The cardinality of a finite set is simply the number of elements in that set. We use the notation $|X|$ to denote the cardinality of some set X. We have already seen by theorem 7 that $|\mathbb{R}| > |\mathbb{N}|$, and the Continuum Hypothesis is related to this relation on the cardinals. We denote the cardinal number of the natural numbers $\mathbb{N}$ to be $\aleph_0$, and the cardinal number of the real numbers $\mathbb{R}$ to be $\aleph_1$.

Theorem 9. *(Continuum Hypothesis) There does not exist a set c such that $|\mathbb{N}| < |c| < |\mathbb{R}|$*

Another way of stating the Continuum Hypothesis in the language of before is $2^{\aleph_0} = \aleph_1$, where 2^n represents the size of the power set of n, because a real number can be written as a sequence of integers via its decimal expansion.

The Continuum Hypothesis is actually independent of the axioms of mathematics, meaning there is a consistent model of mathematics where it holds, and a consistent model of mathematics where its negation holds.

An alternate generalization of the natural numbers to infinity is the Ordinals.

Definition 10. *An **ordinal** is the sizes of a well ordered set.*

We first must understand what it means to be well ordered.

Definition 11. *A set is **well ordered** if it obeys the 3 properties*

(1) *Transitivity: if $x \leq y$ and $y \leq z$ then $x \leq z$*
(2) *Reflexivity: $x \leq x$*
(3) *Antisymmetry: if $x \leq y$ and $y \leq x$ then $x = y$*

For some examples of ordinals, we have $0 = \{\}$, $1 = \{0\}$, $2 = \{0, 1\}$ etc. In general, the successor of a, called a^+, is defined as $a^+ = \{a\} \cup a$ This goes on until $\omega = \{0, 1, 2, 3, \cdots\}$. We can repeat this process to get $\omega + 1 = \{0, 1, 2, 3 \cdots, \omega\}$. This is a key difference between cardinals and ordinals. ω and $\omega + 1$ have the same cardinality because there is a bijection between then (e.x.

$f : \omega \to 0$, $f : x \to x + 1$) but there does not exist an order-preserving bijection, so they have different ordinalities. This process can continue for $\omega + 1, \omega + 2, \omega + 3, \cdots, \omega + \omega$ and so on. The ordinals challenge our intuitions about infinities, but are a very fun topic.

Using the cardinal numbers, one interesting theorem is Konig's Theorem.

Theorem 12. *Konig's Theorem: If I is a (possibly infinite) set, and $|A_i| < |B_i|$ for all $i \in I$, then $|\sum_{i\in I} A_i| < |\prod_{i\in I} B_i|$, where $\sum_{i\in I} A_i$ denotes the disjoint union and $\prod_{i\in I} B_i$ denotes the Cartesian product.*

Proof: By the definition of $<$, we have an injective map $f_i : A_i \to B_i$. We are going to construct an injective map $f : \sum_{i\in I} \to \prod_{i\in I} B_i$.

Choose some points $x_j \in B_j \setminus A_j$. Let $\pi_j : \prod_i B_i \to B_j$ be the product projection (i.e. a projection is taking out the jth element from a product) and let $i_j : A_j \to \sum_i A_i$ the coproduct inclusion (i.e. an inclusion is just taking the union with x_j), define a map f: $\sum_j A_j \to \prod_j B_j$:

f(x) = $\begin{cases} (\pi_j \circ f \circ i_k)(a) & = f_j(a) \text{ if } a \in A_k \text{ and } j = k \\ & = x_j \text{ if } a \in A_k \text{ and } j \neq k \end{cases}$

One can prove that f is injective, and so $|\sum_i A_i| \leq |\prod_i B_i|$.

5. Easton's Theorem

The generalization of the statement that the Continuum Hypothesis is independent of the axioms of mathematics is Easton's Theorem, which generates a whole set of related statements, all of which are independent of the axioms of mathematics. First we need some necessary definitions.

Definition 13. *A set is **partially ordered** if it has an ordering relationships $\leq$ on the elements, and this relationship obeys the 3 laws of defintion 11.*

For example, the real numbers along with $\leq$ form a partially ordered set, as do the power set $\mathcal{P}(X)$ along with $\subseteq$. The next definition we need is that of Cofinality.

Definition 14. *Given a partially-ordered set Q, the **cofinality** of Q is the largest cardinal number κ such that every function f:[κ]$\to$Q (where [κ] is any set of cardinality κ) has an upper bound.*

In other words an element x of Q such that if y belongs to the image of f then $y < x$. An example is that the cofinality of $\aleph_0$ (the cardinal number of $\mathbb{N}$) is $\aleph_0$. This is because any function $f : \mathbb{N} \to \mathbb{N}$ cannot reach arbitrarily high.

Theorem 15. *Konig's Corollary: For any infinite cardinal κ, the cardinal 2^κ has cofinality greater than κ.*

Konig's Corollary is related to confinality, and it follows as a consequence from Konig's Theorem (see theorem 12). Proof: κ has the same cardinality as $\kappa \times \kappa$, so $2^\kappa = 2^{\kappa\times\kappa} = (2^\kappa)^\kappa$. Let $(\lambda_\alpha)_{\alpha<\kappa}$ be an increasing sequence with least upper bound 2^κ, then we have surjection $f : \sum_{\alpha<\kappa} \lambda_\alpha \to 2^\kappa = (2^\kappa)^\kappa$. Since $2^\kappa = \prod_{i\in\kappa} 2$, we have $f : \sum_{\alpha<\kappa} \lambda_\alpha \to \prod_{i\in\kappa} 2$, but since $2 < \lambda_a$, this contradicts Konig's Theorem. Thus, 2^κ must have a greater cardinality than κ.

Theorem 16. *Easton's Theorem: If $f : \kappa \to \kappa$ is a function on the cardinals such that*

(1) *f is increasing (preserves the order of $\leq$)*
(2) *κ is less than the confinality of $f(\kappa)$. (see definition 14)*

Then there is a model of ZFC such that $2^\kappa = f(\kappa)$.

Easton proved this theorem using forcing. Note that CH is a special case of Easton's Theorem, in the case that $f(\kappa) = 2^\kappa$ It is a generalization of CH that says that not only is CH independent of ZFC, but describes an infinite family of statements that are all independent of ZFC (see [Eas64]).

6. Equivalent Statements

One consequence of the Continuum Hypothesis is that many equivalent statements keep appearing. One example is Wetzel's Problem, covered in PFTB. Another example is Freiling's axiom of symmetry.

Axiom 17. *(Freiling) Let $f : x \to A_x$ be a function from the real numbers $x \in [0, 1]$ to the countable subsets of such reals $A \subset [0, 1]$. Then, there exist two real numbers x and y such that $x \notin f(y)$ and $y \notin f(x)$.*

Frieling's intuition for this was to imagine throwing 2 darts at the real number line, and they land at x and y respectively. The number y is almost certainly not in $f(x)$ because $f(x)$ is countable and $[0, 1]$ is uncountable.

Theorem 18. *CH is equivalent to the negation of Frieling's axiom*

Proof: (Forward direction). Suppose $2^{\aleph_0} = \aleph_1$. Then there exists a bijection $\sigma : \mathbb{N} \to \wp(\mathbb{N})$, where $\wp(S)$ denotes the power set of some set S. Define $f : \wp(\mathbb{N}) \to \wp(\wp(\mathbb{N}))$ via $f : \sigma(\alpha) \mapsto \{\sigma(\beta) : \beta \preceq \alpha\}$. The function f satisfies the requirements of Frieling's Axiom, yet there do not exist 2 numbers x,y by Frieling's axiom because one of x,y must be less than the other, so if $x < y$, then $x \in f(y)$, contradicting Frieling's Axiom.

(Backward direction) Suppose Frieling's axiom doesn't hold. Then, choose some f that satisfies the requirements of Frieling's axiom. Define an ordering relation on $\wp(\mathbb{N})$, call it $\leq_f$, where $A \leq_f B$ if $A \in f(B)$. Define a strictly increasing chain of sets $(A_\alpha \in \wp(\kappa))_{\alpha<\kappa^+}$ as follows: at each stage choose $A_\alpha \in \wp(\kappa) \setminus \bigcup_{\beta<\alpha} f(A_\beta)$. This sequence is cofinal in the order defined, i.e. every member of $\wp(\kappa)$ is $\leq_f$ some A_α. Thus we may define a map $g : \wp(\kappa) \to \kappa^+$ by $B \mapsto \min\{\alpha < \kappa^+ : B \in f(A_\alpha)\}$. So $\wp(\kappa) = \bigcup_{\alpha<\kappa^+} g^{-1}\{\alpha\} = \bigcup_{\alpha<\kappa^+} f(A_\alpha)$ which is union of κ^+ many sets each of size $\leq \kappa$. Thus $2^\kappa \leq \kappa^+ \cdot \kappa = \kappa^+$ and we are done because this is equivalent to CH.

7. Conclusion

While the case can be made for either CH or $\neg$ CH being the better model for set theory, another view is the multiverse view. In the multiverse view there are many models of set theory, but no one "true" model. The various models are all equally valid, but some are more useful or than others.

In the universe view the continuum hypothesis is a meaningful question that is either true or false, but in the multiverse view, CH depends on the model of set theory selected. This is one way to "resolve" the CH debate.

References

[Eas64] William Bigelow Easton. *Powers of regular cardinals*. Princeton University, 1964.